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q-Difference Equations.

By Rev. F. H. Jackson.

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§ 1.

The properties of the q-hypergeometric series

$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q\,)(1-q^{\gamma})}x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q\,)(1-q^{2}\,)(1-q^{\gamma})(1-q^{\gamma+1})}x^{2}+\dots.$$

have been investigated by Heine (*Kugelfunctionen*, ed. 1878, pp. 99 et seq.); also by Thomæ (*Crelle's Journal*, Vol. LXX). In this note I wish to introduce a notation which will, I hope, be of utility in developing the theory of q-functions generally: For example, if we replace the differential operative symbol ϑ in the differential equation of the hypergeometric series $F(\alpha, \beta, \gamma, x)$ by

$$[\vartheta]=(q^{\vartheta}-1)/(q-1),$$

we have

$$x[\vartheta+\alpha][\vartheta+\beta]y-[\vartheta][\vartheta+\gamma-1]y=0, \tag{1}$$

a q-difference equation with particular solutions corresponding in every detail to the solutions of

$$x(\vartheta+\alpha)(\vartheta+\beta)y-\vartheta(\vartheta+\gamma-1)y=0.$$

If we define Δ, a symbol of operation, by the equation

$$\Delta\Phi(x)=\frac{\Phi(qx)-\Phi(x)}{qx-x}, \tag{2}$$

then symbolically, in terms of the differential operator $\left(\vartheta=x\,\dfrac{d}{dx}\right)$,

$$\begin{aligned} x\Delta &= \frac{q^{\vartheta}-1}{q\;-1} \\ &= [\vartheta]. \end{aligned}$$

Repetition of (2) gives

$$\Delta^{2}\Phi(x)=\frac{\Phi(q^{2}x)-(q+1)\Phi(qx)+q\Phi(x)}{qx^{2}(q-1)^{2}},$$

which is symbolically

$$x^2\Delta^2 = \frac{q^{2\vartheta} - (q+1)q^{\vartheta} + q}{q(q-1)^2} \qquad x^2\Delta^2 = [\vartheta][\vartheta - 1],$$

and in general,

$$x^n\Delta^n = [\vartheta][\vartheta - 1] \ldots [\vartheta - n + 1]. \tag{3}$$

We see that in case the functions on which we operate are differentiable in the range of the variable, we may write

$$\lim_{q \longrightarrow 1} [\vartheta] = \vartheta = x\frac{d}{dx},$$

$$\lim_{q \longrightarrow 1} [\vartheta][\vartheta - 1] \ldots . [\vartheta - n + 1]$$
$$= \vartheta(\vartheta - 1) \ldots . (\vartheta - n + 1) = x^n \frac{d^n}{dx^n} \quad \text{(Boole).} \tag{4}$$

§ 2.

The well-known and very useful operational theorem

$$F(\vartheta)x^m v = x^m F(\vartheta + m)v \tag{5}$$

has its counterpart in

$$F[\vartheta]x^m v = x^m F[\vartheta + m]v\,;$$

for

$$\begin{aligned}[\vartheta]x^m v_x &= (q^m x^m v_{qx} - x^m v_x)/(q-1)\\ &= x^m(q^m v_{qx} - v_x)/(q-1)\\ &= x^m(q^{m+\vartheta}v - v)/(q-1)\\ &= x^m[\vartheta + m]v\,;\end{aligned}$$

therefore

$$[\vartheta]^r x^m v = x^m[\vartheta + m]^r v.$$

This equation holds good when r is a negative integer; for, on writing

$$[\vartheta + m]^r v = V, \qquad v = [\vartheta + m]^{-r} V,$$

we have

$$[\vartheta]^r x^m [\vartheta + m]^{-r} V = x^m V;$$

on operating with $[\vartheta]^{-r}$, we obtain

$$x^m[\vartheta + m]^{-r} V = [\vartheta]^{-r} x^m V;$$

so that, finally, if $F(a)$ denotes an algebraic function of a, we have symbolically

$$F([\vartheta])x^m v = x^m F([\vartheta + m])v.$$

(COROLLARY). It will follow that if U be any function of the form $A + Bx + Cx^2 + \ldots .,$

$$\frac{1}{\psi[\vartheta]} U = \frac{A}{\psi[0]} + \frac{B}{\psi[1]}x + \frac{C}{\psi[2]}x^2 + \ldots . \tag{6}$$

§ 3.

Before proceeding to solve any q-difference equations, I remark that if the reverse operation Δ^{-1} be named q-integration, and we use the symbolism

$$\Delta^{-1}\phi(x) \equiv \mathop{\mathrm{S}} \Phi(x)d(qx), \tag{7}$$

remembering that subject to conditions of integrability

$$\lim_{q \longrightarrow 1} \mathop{\mathrm{S}} \Phi(x)d(qx) = \int \Phi(x)dx, \tag{8}$$

then the q-hypergeometric difference equation will have a solution

$$\mathop{\mathrm{S}}\limits_0^1 u^{\beta-1}(1-qu)^{\gamma-\beta-1}(1-q^{\alpha}ux)^{-\alpha}d(qu)$$

reducing in the limit ($q \longrightarrow 1$) to

$$\int u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^{-\alpha}du.$$

Similarly, for the q-series

$$F_q([\alpha][\beta][\gamma][\delta][\varepsilon]\,;\,x),$$

and equation

$$x[\vartheta+\alpha][\vartheta+\beta][\vartheta+\gamma]y - [\vartheta][\vartheta+\delta-1][\vartheta+\varepsilon-1]y = 0,$$

we have an integral

$$\mathop{\mathrm{S}}\limits_0^1 u^{\beta-1}(1-qu)^{\delta-\beta-1}v^{\gamma-1}(1-qu)^{\varepsilon-\gamma-1}(1-q^{\alpha}uvx)^{-\alpha}d(qu)d(qv).$$

Of course $(1-qu)^m$ denotes the q-binomial product

$$\prod_{r=1}^{\infty} \frac{(1-q^r u)}{(1-q^{n+r}u)} = 1 - q[n]u + q^3\frac{[n][n-1]}{[2]!}u^2 - \cdots\cdots \qquad (|q|<1).$$

§ 4.

q-Finite Integration.

It is well-known (Bromwich, "Theory of Infinite Series," pp. 418–419)

$$\int_0^b f(x)dx = \lim_{q \longrightarrow 1} b(1-q)\{f(b) + qf(bq) + q^2 f(bq^2) + \cdots\cdots\}.$$

Consider

$$\Delta u = \phi(x),$$

which is

$$\frac{1}{x}[\vartheta]u = \phi(x),$$

$$\begin{aligned} u &= \frac{(1-q)}{(1-q^{\vartheta})}x\phi(x) + \frac{(1-q)}{(1-q^{\vartheta})}\cdot 0 \\ &= (1-q)\{1 + q^{\vartheta} + q^{2\vartheta} + \dots\}x\phi(x) + C, \end{aligned}$$

$$\mathop{\mathrm{S}}\limits_{0}^{b}\phi(x)d(qx) = b(1 - q\{\Phi(b) + q\Phi(qb) + q^2\Phi(q^2b) + \dots\}. \tag{9}$$

C will be in general any function of multiplicative periodicity q. Such a function, for example, as

$$exp\left(i\frac{\log x}{\log q}\right).$$

If in (9) we expand $(1-q)/(1-q^{\vartheta})$ in powers of ϑ, using Bernoulli's coefficients, we find

$$\begin{aligned} \mathop{\mathrm{S}}\limits_{0}^{b}\Phi(x)d(qx) &= \frac{(q-1)}{\log q}\int_0^b \phi(x)dx + \tfrac{1}{2}(1-q)b\Phi(b) \\ &\quad + (1-q)\sum_{r=1}^{\infty}(-1)^r\frac{B_{2r-1}}{2r\,!}(\log q)^{2r-1}(\vartheta)^{2r-1}x\phi(x)\Big]_{x=b} \end{aligned} \tag{10}$$

Now $\lim\limits_{q=1}\frac{q-1}{\log q} = 1$, so that in this form we see in detail how

$$\int_0^b \Phi(x)dx = \lim_{q\longrightarrow 1} b(1-q)\{\Phi(b) + q\Phi(qb) + \dots\}. \tag{11}$$

§ 5.

Solutions of Equations in Series.

Since

$$[\vartheta + \alpha]x^n = \frac{q^{\vartheta+\alpha}-1}{q-1}x^n = \frac{q^{\alpha}q^{\vartheta}x^n - x^n}{q-1} = \frac{(q^{\alpha+n}-1)x^n}{(q-1)} = [\alpha + n]x^n,$$

$$[\vartheta + \alpha][\vartheta + \beta]x^n = [n + \alpha][n + \beta]x^n$$

and

$$\begin{aligned} [\vartheta + \alpha]x^n \log x &= \frac{q^{\vartheta+\alpha}x^n \log x - x^n \log x}{q-1} \\ &= \frac{q^{\alpha}q^{n}x^n \log qx - x^n \log x}{q-1} = [\alpha + n]x^n \log x + \frac{\log q}{q-1}x^n q^{\alpha+n}, \end{aligned}$$

$$\begin{aligned} [\vartheta + \alpha][\vartheta + \beta]x^n \log x &= [\alpha + n][\beta + n]x^n \log x + \lambda q^{\alpha+n}[\beta + n]x^n \\ &\quad + \lambda q^{\beta+n}[\alpha + n]x^n \qquad (\lambda = \log q/(q-1)), \end{aligned} \tag{12}$$

it is obvious that if we substitute the series

$$A_0x^\theta + A_1x^{\theta+1} + A_2x^{\theta+2} + \dots .$$

for y in the equation

$$x[\vartheta + \alpha][\vartheta + \beta]y - [\vartheta][\vartheta + \gamma - 1]y = 0,$$

we obtain an indicial equation

$$[\theta][\theta + \gamma - 1] = 0$$

with two principal roots $\theta = 0,\ 1 - \gamma$, and the usual infinity of roots involving the logarithm. Taking

$$A_n = A_0\frac{[n - 1 + \alpha + \theta]\dots[\alpha]\,.\,[n - 1 + \beta + \theta]\dots[\beta]}{[n + \theta]\dots[1 + \theta]\,.\,[n - 1 + \gamma + \theta]\dots[\gamma]},$$

we find for the root $\theta = 0$, the solution

$$F_q([\alpha][\beta][\gamma],\, x),$$

and for the root $\theta = 1 - \gamma$ a particular solution

$$x^{1-\gamma}F_q([\alpha - \gamma + 1][\beta - \gamma + 1]\,[2 - \gamma]\,;\, x).$$

In case $1 - \gamma$ be a positive integer, say p, we substitute for y the series

$$\sum_{r=0}^{\infty}(A_r + B_r \log x)x^{r+\theta}.$$

Now if $\Pi_q[m]$ denote the q-Π function analogous to Gauss' Π function,

$$\Pi_q[m] = [m]\Pi_q[m - 1].$$

If $\frac{d}{dx}\log \Pi_q([m])$* be denoted Ψ_m, we have

$$\Psi_m = \lambda\frac{q^m}{[m]} + \Psi_{m-1},$$

where $\lambda = \log q/(q - 1)$. The substitution of

$$\sum(A_r + B_r \log x)x^{\theta+r} \quad \text{in} \quad x[\alpha + \vartheta][\beta + \vartheta]y - [\vartheta][\vartheta - p]y = 0$$

gives an indicial equation $[\theta][\theta - p] = 0$; and if $\sum$ is to be a solution,

$$\begin{aligned}\sum_{r=0}^{\infty}\{A_r[\alpha + \theta + r][\beta + \theta + r]x^{\theta+r+1} &\quad - \quad A_r[\theta + r][\theta - p + r]x^{\theta+r},\\ B_r[\alpha + \theta + r][\beta + \theta + r]x^{\theta+r+1}\log x &- \quad B_r[\theta + r][\theta - p + r]x^{\theta+r},\\ \lambda B_r[\alpha + \theta + r]q^{\beta+\theta+r}x^{\theta+r+1} &\quad - \lambda B_r[\theta + r]q^{\theta+r-p}x^{\theta+r},\\ \lambda B_r[\beta + \theta + r]q^{\alpha+\theta+r}x^{\theta+r+1} &\quad - \lambda B_r[\theta - p + r]q^{\theta+r}x^{\theta+r}\}\end{aligned}$$

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* *Proc. R. S. Lond.*, Vol. LXXVI A, p. 130.

must vanish identically. Hence

$$A_{r+1}[\theta + r + 1][\theta - p + r + 1] \\ = A_r[\theta + \alpha + r][\theta + \beta + r] \qquad (r = 0, 1, \ldots, p-1),$$

$$A_p, A_{p+1}, \ldots = 0.$$

So putting $\theta = 0$, and writing $[\alpha][\alpha+1]\ldots.[\alpha+n-1] = [\alpha]_n$, we find a solution

$$\frac{[\alpha]_p[\beta]_p}{[p]!\,[1-p][2-p]\ldots.[-1]} x^p F_q([\alpha+p][\beta+p][1+p], x) \log x$$

$$+ \sum_{n=0}^{p=1} \frac{[\alpha]\ldots.[\alpha+n-1]\,.\,[\beta]\ldots.[\beta+n-1]}{[n]!\,[1-p][2-p]\ldots.[n-p]} x^n$$

$$+ \sum_{n=p}^{\infty} \frac{[\alpha]_n[\beta]_n}{[n]!\,[n-p]!\,[1-p][2-p]\,.\,[-1]} x^n \Phi_n, \tag{13}$$

where $\Phi_n = \Psi[n-1+\alpha) + \Psi[n-1+\beta] - \Psi[n] - \Psi[n-p]$. We see that

$$\Phi_n - \Phi_{n-1} = \frac{q^{n-1+\alpha}}{[n-1+\alpha]} + \frac{q^{n-1+\beta}}{[n-1+\beta]} - \frac{q^n}{[n]} - \frac{q^{n-p}}{[n-p]}.$$

The solution, in case $1-\gamma$ be a negative integer, offers no difficulty, and in fact we can use a slight modification of Frobenius' method to obtain these forms of integrals. Simply substituting $\sum c_r x^{\theta+r}$ in the q-difference equation, choosing

$$c_n = c_0 \frac{[n-1+\alpha+\theta]\ldots.[\alpha][n-1+\beta+\theta]\ldots.[\beta]}{[n+\theta]\,.\,[1+\theta]\,.\,[n-p+\theta]\ldots.[1-p+\theta]},$$

make $c_0 = C[\theta]$, so that

$$y\phi(\vartheta)y = C[\theta]^2[\theta-p]x^\theta,$$

$$y = C\sum_{n=0} \frac{[\alpha+\theta]_n[\beta+\theta]_n}{[1+\theta]_n[1-p+\theta]_n} [\theta] x^{\theta+n}.$$

Two particular integrals will be given by

$$\{y\}_{\theta=0} \text{ and } \left\{\frac{dy}{d\theta}\right\}_{\theta=0}.$$

The first gives a constant multiple of $x^p F_q([\alpha+p][\beta+p][1+p], x)$. The second, after making $C = 1$, reduces to (13), given above. In case $1-\gamma$ is a negative integer, $-p$, we write $c_0 = [\theta+p]C$ and find two particular solutions given by

$$\{y\}_{\theta=-p}; \quad \left\{\frac{dy}{d\theta}\right\}_{\theta=-p}$$

The first of these gives $F_q([\alpha][\beta][1+p], x)$; the second gives

$$\frac{[\alpha-1]\,.\,[\alpha-p]\,.\,[\beta-1]\,..\,[\beta-p]}{[p]\,!\,[1-p]\,..\,[-1]}\,F_q([\alpha][\beta][1+p], x)\log x$$

$$+\sum_{n=0}^{p-1}\frac{[\alpha-p]_n[\beta-p]_n}{[n]\,!\,[n-q]\,.\,[1-q]}\,x^{n-p}$$

$$+\sum_{n=p}^{\infty}\frac{[\alpha-p]_n[\beta-p]_n}{[n]\,!\,[n-q]\,!\,[1-q][2-p]\,.\,[-1]}\,x^{n-p}\Phi_n, \tag{14}$$

$$\Phi_n=\psi[n-1+\alpha-p]+\psi[n-1+\beta-p]-\psi[n]-\psi[n-p].$$

§ 6.

The General q-Hypergeometric Series.

$$F_q([\alpha_1]\,\ldots.\,[\alpha_r],\,[\beta_1]\,[\beta_2]\,\ldots.\,[\beta_s],\,x),$$

$$x^{1-\beta_m}F_q[(\alpha_1-\beta_m+1]\ldots.[\alpha_r-\beta_m+1],$$
$$[2-\beta_m][\beta_1-\beta_m+1]\,..\,[\beta_s-\beta_m+1],\,x),$$

where $m=1, 2, 3, 4, \ldots., s$ and $1-\beta_m$ is not zero or an integer, are solutions of

$$x[\vartheta+\alpha_1][\vartheta+\alpha_2]\,..\,[\vartheta+\alpha_r]y-[\vartheta][\vartheta+\beta_1-1]\,..\,[\vartheta+\beta_s-1]y=0. \tag{15}$$

§ 7.

Solution of $[\vartheta+k]\,y=V$ *(* V *a given function of* x*).*

We transform this equation into

$$y=[\vartheta+k]^{-1}V,$$

which by (5) becomes

$$y=x^{-k}[\vartheta]^{-1}x^kV.$$

Expanding $[\vartheta]^{-1}$ in powers of ϑ,

$$y=C+\frac{(q-1)}{\log q}x^{-k}\int Vx^{k-1}dx-(q-1)\frac{V}{2}$$
$$+(q-1)\sum_{r=1}^{\infty}(-1)^{r-1}\frac{B_{2r-1}}{2r\,!}(\log q)^{2r-1}x^{-k}\left(x\frac{d}{dx}\right)^{r-1}x^kV.$$

The solution of

$$[\vartheta+\alpha_1][\vartheta+\alpha_2]\ldots.[\vartheta+\alpha_n]y=V \tag{16}$$

by resolution of the inverse operator $\{[\vartheta+\alpha_1]\ldots.[\vartheta+\alpha_n]\}^{-1}$ into partial fractions is obvious. Now an equation

$$a_2\Phi(q^2x)+a_1\Phi(qx)+a_0\Phi(x)=V,$$

in which V is a given function of x and the form Φ is to be determined, is equivalent to finding a solution of

$$[\vartheta+\alpha][\vartheta+\beta]y=V.$$

In general, such an equation as (16), given above, corresponds to

$$a_n\Phi(q^nx)+a_{n-1}\Phi(q^{n-1}x)+\ldots\ldots+a_1\Phi(qx)+a\Phi(x)=V,$$

and Φ can be determined provided the factors of the polynomial

$$a_nr^n+a_{n-1}r^{n-1}+\ldots\ldots+a_1r+a_0$$

are known.

§ 8.

Definite Integral Solutions.

To determine a q-finite integral form satisfying the difference equations of the q-hypergeometric series we proceed as follows: Let

$$(1-au)^{(n)}=\prod_{r=p}^{\infty}\frac{(1-auq^r)}{(1-auq^{r+n})},$$

we note that

$$\Delta(1-au)^{(n)}=-a[n](1-qau)^{(n-1)},$$

so that, using the formula for q-part integration, we have

$$\mathop{\mathrm{S}} u^{m-1}(1-qu)^{(n-1)}=-\frac{1}{[n]}u^{m-1}(1-u)^{(n)}+\mathop{\mathrm{S}}\frac{[m-1]}{[n]}u^{m-2}(1-qu)^{(n)}.$$

Now in case n be positive, $(1-u)^{(n)}$ vanishes when $u=1$, so that we have

$$\mathop{\mathrm{S}}\limits_0^1 u^{m-1}(1-qu)^{(n-1)}=\frac{[m-1]}{[n]}\cdot\mathop{\mathrm{S}}\limits_0^1 u^{m-2}(1-qu)^{(n)},$$

which serves to indicate a function $B_q([m],[n])$ corresponding to Euler's Beta Function. If we expand $(1-qu)^{(n-1)}$ as

$$1-[n-1]qu+\frac{[n-1][n-2]}{[2]!}q^3u^3+\ldots\ldots+(-1)^r\frac{[n-1]\ldots\ldots[n-r]}{[r]!}q^{\frac{1}{2}r(r+1)}u^r+\ldots\ldots$$

and take the q-integral

$$\mathop{\mathrm{S}}\limits_0^1 u^{m-1}\{1-[n-1]qu+\ldots\ldots\}d(qu)$$

term by term, we get

$$\frac{1}{[m]}\left\{1-\frac{[n-1][m]}{[m+1][1]}q+\frac{[n-1][n-2][m][m+1]}{[m+1][m+2][2]!}q^3-\ldots\ldots\right\}$$

$$=\frac{1}{[m]}\left\{1+\frac{[1-n][m]}{[m+1][1]}q^n+\frac{[1-n][2-n][m][m+1]}{[m+1][m+2][2]!}q^{2n}+\ldots\ldots\right\},$$

$$= \frac{1}{[m]} \prod_{r=1}^{\infty} \frac{[m+n+r]}{[n+r][m+r+1]}$$

$$= \prod_{r=1}^{\infty} \frac{[m+n+r]}{[n+r]\,[m+r]}$$

$$= \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)},$$

where $\Gamma_q(m)$ is the basic or q-gamma function* whose difference equation is

$$\Gamma_q(n+1) = [n]\Gamma_q(n).$$

The above is a special case (for $\alpha = 1 - n$, $\beta = m$, $\gamma = m + 1$) of the well-known q-hypergeometric series

$$F([\alpha]\,[\beta]\,[\gamma] q^{\gamma-\alpha-\beta}),$$

the sum of which is

$$\frac{\Gamma_q(\gamma-\alpha-\beta)\Gamma_q(\gamma)}{\Gamma_q(\gamma-\alpha)\Gamma_q(\gamma-\beta)}$$

If now we consider

$$\mathop{\mathrm{S}}_{0}^{1} u^{\beta-1}(1-qu)^{\gamma-\beta-1}(1-q^{\alpha}xu)^{-\alpha}d(qu)$$

and integrate after replacing $(1 - q^{\alpha}xu)^{-\alpha}$ by its expansion

$$1 + [\alpha]x + \frac{[\alpha][\alpha+1]}{[2]\,!}x^2 + \ldots.,$$

we find that this integral is

$$\frac{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}{\Gamma_q(\gamma)} F_q([\alpha][\beta][\gamma], x), \tag{17}$$

provided β and $\gamma - \beta$ are both positive.

Similarly, the double q-integral

$$\mathop{\mathrm{S}}_{0}^{1}\mathop{\mathrm{S}}_{0}^{1} u^{\beta-1}(1-qu)^{\delta-\beta-1}\,v^{\gamma-1}(1-qv)^{\epsilon-\gamma-1}(1-q^{\alpha}uvx)^{-\alpha}d(qu),$$

if we expand $(1 - q^{\alpha}uv)^{-\alpha}$ and integrate term by term the series which is absolutely convergent if $|q| < 1$, becomes

$$\mathop{\mathrm{S}}_{0}^{1} u^{\beta-1}(1-qu)^{\delta-\beta-1}\frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)}{\Gamma_q(\varepsilon)} F_q([\alpha][\gamma][\varepsilon], ux)d(qu). \tag{18}$$

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Replacing $F_q([\alpha][\gamma][\varepsilon], ux)$ by the infinite series and taking the q-integrals term by term, we have for the general term

$$\frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)}{\Gamma_q(\varepsilon)} \int_0^1 \frac{[\alpha]_r[\gamma]_r}{[\varepsilon]_r} x^r u^{\beta+r-1}(1-qu)^{\delta-\beta-1} d(qu),$$

$$= \frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)[\alpha]\ldots.[\alpha+r-1][\gamma]\ldots[\gamma+r-1]}{\Gamma_q(\varepsilon)[\varepsilon]\ldots.[\varepsilon+r-1]} \times \frac{\Gamma_q(\beta+r)\Gamma_q(\delta-\beta)}{\Gamma_q(\delta+r)} x^r;$$

so that (18)

$$= \frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)\Gamma_q(\beta)\Gamma_q(\delta-\beta)}{\Gamma_q(\varepsilon)\Gamma_q(\delta)} F_q([\alpha][\beta][\gamma], [\delta][\varepsilon], x),$$

provided β, $\delta-\beta$, γ, $\varepsilon-\gamma$ be all positive quantities.

§ 9.

In conclusion, it may be of interest to note that

$$[\vartheta+n][\vartheta-n]y + x^2 y = 0$$

is the q-equation satisfied by a function

$$J_n(q, x) = \frac{x^n}{[2][4]\ldots[2n]}\left\{1 - \frac{x^2}{[2][2n+2]} + \frac{x^4}{[2][4][2n+2][2n+4]} - \ldots.\right\}.$$

In case n be not a positive integer, $[2][4]\ldots[2n]$ may be replaced by a suitable q-gamma function.

$$[\vartheta+m+n][\vartheta+m-n][\vartheta-m+n][\vartheta-m-n]y + [2\vartheta][2\vartheta-2]x^2 y = 0$$

is the equation satisfied by

$$J_m(q, x) \times J_n\left(\frac{1}{q}, x\right)$$

$$= \sum_{r=0}^{r=\infty} (-1)^r \frac{\{2m+2n+4r\}!}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!} x^{m+n+2r}.$$

The theory of these functions, apart from the q-difference equations which they satisfy, has been discussed (*Proc. L. M. S.*, Ser. 2, Vol. II, Part III; Vol. III, Part I).

We note that

$$\{2m\}! = [2][4]\ldots[2m] \text{ and } J_n\left(\frac{1}{q}, x\right) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{n+2r}}{\{2n+2r\}!\{2r\}!} q^{n^2+2r(n+r)}.$$